Int. J. Solids Structures, 1977, Vol. 13, pp. 39–51. Pergamon Press. Printed in Great Britain.

SIMULATION OF ATTITUDE MOTIONS OF TORQUE-FREE, DEFORMABLE SPACECRAFT†

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(Received 8 July 1976)

Abstract—Equations of motion are derived for torque-free, elastic, dissipative systems possessing a finite number of degrees of freedom. The equations are linearized about a particular solution in the variables describing the "internal" configuration of the system, and then SAM, a computer program based on these equations, is described. The program permits the user to obtain information about motions of a given system without writing any computer code. The paper concludes with an illustrative example involving a comparison of results obtained, on the one hand, by solving the exact equations of motion and, on the other hand, by using SAM to explore the behavior of a satellite containing a passive nutation damper. The example shows rather vividly both that relatively small deformational motions can give rise to large attitude motions and that SAM can describe these motions satisfactorily.

INTRODUCTION

Computer simulations of attitude motions of deformable spacecraft play an important role in spacecraft design and operation. Since it is generally difficult, time consuming, and expensive to produce such simulations, it is worthwhile to seek means to facilitate this task. The present paper is intended to do this by showing how to bring the equations governing a large class of spacecraft motions into a form permitting the construction of a computer program that is particularly easy to use.

All spacecraft are deformable to some extent. In certain situations, the deformations are so small that they have no discernible effect on attitude motions and can, therefore, be left out of account. In other words, such motions can be simulated by working with equations based on a rigid body model of the spacecraft. In other situations, relative motions of parts of a vehicle are so large that a satisfactory simulation can be achieved only by using equations that characterize deformations in all detail. The most frequently encountered situations, however, are those involving deformations which, while relatively small, nevertheless affect or, at least, might affect, attitude motions significantly. The principal idea underlying the present work is that one may hope to simulate such motions satisfactorily by employing an algorithm based on equations describing the deformational motion only approximately. Before further comments are made regarding approximations, a few words about the mathematical model employed in the analysis are in order.

One of the more common ways to model a nonrigid spacecraft is to treat it as a set of N rigid bodies connected to each other in such a way that each body shares at least one point with one other body. If there are no closed loops, such a system is called a tree structure. Since 1965, when Hooker and Margulies[1] first dealt with an N-body tree structure, much attention has been devoted to this subject[2–8], including efforts to produce computer programs[9–11]. Simultaneously, efforts were undertaken to analyze spacecraft that cannot be modeled as tree structures. For example, analyses accommodating contiguous bodies capable of translating relative to each other have been performed[12, 13]; programs have been written to deal with flexible spacecraft members[14–16]; and sets of equations applicable to systems containing axially symmetric rotors have been formulated[17, 18].

The equations and the computer program developed in the present paper apply to any set of particles connected to each other by time-invariant, holonomic constraints. Explicit provisions are made for allowing subsets of the set of particles to form rigid bodies. Hence, tree structures are included among the systems to which this work applies, as are systems involving contiguous bodies capable of translating relative to each other. Since, by resorting to the so-called discrete element method, one can treat continuous elements as sets of particles, attitude motions of spacecraft containing such elements can be studied in terms of the present theory. In this sense,

†This work was supported, in part, by the National Science Foundation under Grant No. NSF-ENG-75-18680 to Stanford University.

the present work is more general than the references cited so far, and it parallels the efforts of a number of authors[19–24] who have dealt with flexible spacecraft. However, the investigation is confined to torque-free systems; cyclic coordinates are ruled out, so that rotors are excluded from consideration; and a linearization is performed, rendering the resulting equations approximate. Thus, there certainly exist problems the solutions of which lie outside of the scope of the program, but which can be solved by using existing programs.

The central feature of the present program is that it can simulate motions accompanied by large attitude changes, despite the fact that it involves linearized equations. The approximations made in order to arrive at these results are similar in character to those employed by Roberson and Likins[25, 26], Larson[27], Fleischer and Likins[11], and Samin and Willems[24].

SYSTEM DESCRIPTION

The system S to be studied consists of ν particles connected to each other by time-invariant, holonomic constraints in such a way that S possesses $n+6$ degrees of freedom in a Newtonian reference frame N. Particles comprising subsets of the particles of S may be connected to each other so as to form rigid bodies.

One can always introduce a set of mutually perpendicular axes B_1, B_2, B_3 originating at S^*, the mass center of S; let B be a reference frame in which B_1, B_2, B_3 are fixed; and describe the configuration of S in N in terms of n non-cyclic "internal" coordinates $q_1, \ldots, q_n$, governing the configuration of S in B, and six "external" coordinates $\theta_1, \ldots, \theta_6$, governing the orientation of B in N and the position of S^* in N. Moreover, this can be done in such a way that, when $q_1 = \cdots = q_n = 0$, B_1, B_2, B_3 are principal axes of inertia of S for S^*. Of the six external coordinates, only three, namely those governing the orientation of B in N, are of interest, and these are designated θ_1, θ_2, θ_3. This is so because it is presumed that S is torque-free, that is, that the resultant moment about S^* of all external forces is equal to zero, so that the orientation of B in N does not depend on the motion of S^* in N.

$\boldsymbol{\omega}$, the angular velocity of B in N, is intimately related to θ_1, θ_2, θ_3 and to their time derivatives, $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$. That is, if $\boldsymbol{\omega}$ is written as

$$\boldsymbol{\omega} = \omega_1\mathbf{b}_1 + \omega_2\mathbf{b}_2 + \omega_3\mathbf{b}_3 \tag{1}$$

where $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ form a dextral set of orthogonal unit vectors respectively parallel to B_1, B_2, B_3, then ω_1, ω_2, ω_3 may be regarded as functions of θ_1, θ_2, θ_3, $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$, and t, the time.

Regarding forces of interaction between particles of S, it is presumed that these are such that the generalized active forces $F_1, \ldots, F_n$ associated with the internal coordinates can be expressed as

$$F_r = -V(q)_{,r} + D_r(q, \dot{q}) \quad r = 1, \ldots, n \tag{2}$$

where $V(q)$, which has the character of a potential function, denotes a function of $q_1, \ldots, q_n$, whereas $D_r(q, \dot{q})$, which is associated with energy dissipation, is a function of $q_1, \ldots, q_n$, $\dot{q}_1, \ldots, \dot{q}_n$ such that

$$D_r(q, 0) = 0 \quad r = 1, \ldots, n \tag{3}$$

In (2), as in the sequel, a subscript comma followed by one or more letters signifies partial differentiations with respect to internal variables when the letter is r, s, or u, and with respect to the time derivatives of the internal variables when the letter has a dot over it, that is, $\dot{r}$, $\dot{s}$, or $\dot{u}$. For example, if f is explictly a function of the external and internal coordinates and of the time derivatives of the internal coordinates, then

$$f_{,rs\dot{u}} = \frac{\partial^3 f}{\partial q_r \partial q_s \partial \dot{q}_u} \quad r, s, u = 1, \ldots, n \tag{4}$$

All differentiations of vectors are performed in reference frame B, except as noted. Furthermore, the summation convention will be used; that is, a repeated index takes on all values in its natural range (1, 2, 3 for i, j, k, l, or m, and $1, \ldots, n$ for r, s, or u).

KINETIC ENERGY AND ANGULAR MOMENTUM

One can generate equations of motion for S by making use of T, the kinetic energy of S in N, and $\mathbf{H}$, the angular momentum of S relative to S^* in N. To this end, one can begin by expressing T as

$$T = 2^{-1}\langle m\mathbf{v}^2\rangle \tag{5}$$

where m and $\mathbf{v}$ stand for the mass and velocity of a generic particle P of S in N, and the symbols $\langle\,\rangle$ denote summation over all particles of S. The velocity $\mathbf{v}$ can be expressed as

$$\mathbf{v} = \mathbf{v}^* + \boldsymbol{\omega} \times \mathbf{p} + \mathbf{p}_{,r}\dot{q}_r \tag{6}$$

where $\mathbf{v}^*$ is the velocity of S^* in N, $\mathbf{p}$ is the position vector of P relative to S^*, and $\mathbf{p}_{,r}$ is the derivative of $\mathbf{p}$ with respect to q_r in B. The first two terms represent the velocity in N of a point fixed in B and instantaneously coincident with P, and the third term represents the velocity of P in B.

Using (6) to eliminate $\mathbf{v}$ from (5) yields

$$\begin{aligned}2T &= \langle m(\mathbf{v}^* + \boldsymbol{\omega} \times \mathbf{p} + \mathbf{p}_{,r}\dot{q}_r)^2\rangle \\ &= \langle m[(\mathbf{v}^*)^2 + 2\mathbf{v}^* \cdot \boldsymbol{\omega} \times \mathbf{p} + 2\mathbf{v}^* \cdot \mathbf{p}_{,r}\dot{q}_r \\ &\quad + (\boldsymbol{\omega} \times \mathbf{p})^2 + 2\boldsymbol{\omega} \times \mathbf{p} \cdot \mathbf{p}_{,r}\dot{q}_r + \mathbf{p}_{,r}\dot{q}_r \cdot \mathbf{p}_{,s}\dot{q}_s]\rangle\end{aligned} \tag{7}$$

Since S^* is the mass center of S, the second and third terms vanish. Now,

$$\begin{aligned}\langle m(\boldsymbol{\omega} \times \mathbf{p})^2\rangle &= \boldsymbol{\omega} \cdot \langle m\mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p})\rangle \\ &= \boldsymbol{\omega} \cdot \langle m(\mathbf{p} \cdot \mathbf{p}\boldsymbol{\omega} - \mathbf{p}\mathbf{p} \cdot \boldsymbol{\omega})\rangle \\ &= \boldsymbol{\omega} \cdot \langle m(\mathbf{p} \cdot \mathbf{p}\mathbf{U} - \mathbf{p}\mathbf{p})\rangle \cdot \boldsymbol{\omega}\end{aligned} \tag{8}$$

where $\mathbf{U}$ is the unit dyadic, and $\langle m(\mathbf{p} \cdot \mathbf{p}\mathbf{U} - \mathbf{p}\mathbf{p})\rangle$ is simply the inertia dyadic $\mathbf{I}$ of S for S^*. Hence

$$\langle m(\boldsymbol{\omega} \times \mathbf{p})^2\rangle = \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \tag{9}$$

and, after defining M_{rs} and $\mathbf{h}_r$ as

$$M_{rs} \triangleq \langle m\mathbf{p}_{,r} \cdot \mathbf{p}_{,s}\rangle \quad r, s = 1, \ldots, n \tag{10}$$

and

$$\mathbf{h}_r \triangleq \langle m\mathbf{p} \times \mathbf{p}_{,r}\rangle \quad r = 1, \ldots, n \tag{11}$$

one has†

$$2T \underset{(7)}{=} \langle m\rangle\mathbf{v}^{*2} + \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} + 2\boldsymbol{\omega} \cdot \mathbf{h}_r\dot{q}_r + M_{rs}\dot{q}_r\dot{q}_s \tag{12}$$

Next, by definition, the angular momentum $\mathbf{H}$ of S in N relative to S^* is given by

$$\begin{aligned}\mathbf{H} &= \langle m\mathbf{p} \times \mathbf{v}\rangle \underset{(6)}{=} \langle m\mathbf{p} \times (\boldsymbol{\omega} \times \mathbf{p} + \mathbf{p}_{,r}\dot{q}_r)\rangle \\ &= \langle m(\mathbf{p} \cdot \mathbf{p}\mathbf{U} - \mathbf{p}\mathbf{p})\rangle \cdot \boldsymbol{\omega} + \langle m\mathbf{p} \times \mathbf{p}_{,r}\dot{q}_r\rangle\end{aligned} \tag{13}$$

Consequently

$$\mathbf{H} = \mathbf{I} \cdot \boldsymbol{\omega} + \mathbf{h}_r\dot{q}_r \tag{14}$$

and (12) and (14) are the desired expressions for kinetic energy and angular momentum.

†Numbers beneath signs of equality refer to corresponding equations.

EQUATIONS OF MOTION

Since, by hypothesis, the resultant moment about S^* of all external forces vanishes, the angular momentum $\mathbf{H}$ of S relative to S^* in N is fixed in N. It follows that

$$\mathbf{n}_i \cdot \mathbf{H} = H_i \quad i = 1, 2, 3 \tag{15}$$

where $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ form a dextral set of orthogonal unit vectors fixed in N and H_1, H_2, H_3 are constants. Moreover, one may require, without loss of generality, that $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ be coincident with $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, respectively, at an arbitrary instant t_0.

If $\mathbf{H}$ is expressed as in (14), then

$$\mathbf{n}_i \cdot (\mathbf{I} \cdot \boldsymbol{\omega} + \mathbf{h}_r \dot{q}_r) \underset{(15)}{=} H_i \quad i = 1, 2, 3 \tag{16}$$

These equations will be referred to as the *external equations of motion.*

By hypothesis, S is holonomic; therefore, the generalized inertia forces $F_1^*, \ldots, F_n^*$ associated with the internal coordinates can be expressed as (see [29], p. 134)

$$F_r^* = -\dot{T}_{,\dot{r}} + T_{,r} \quad r = 1, \ldots, n \tag{17}$$

and, using (12), one obtains

$$\begin{aligned} 2F_u^* \underset{(17)}{=} & \boldsymbol{\omega} \cdot \mathbf{I}_u \cdot \boldsymbol{\omega} - 2(\boldsymbol{\omega} \cdot \mathbf{h}_u + \boldsymbol{\omega} \cdot \mathbf{h}_{u,r} \dot{q}_r) \\ & + 2\boldsymbol{\omega} \cdot \mathbf{h}_{r,u} \dot{q}_r - 2(M_{ur} \ddot{q}_r + M_{ur,s} \dot{q}_r \dot{q}_s) \\ & + M_{rs,u} \dot{q}_r \dot{q}_s \quad u = 1, \ldots, n \end{aligned} \tag{18}$$

Equations of motion can now be formulated by setting sums of generalized active forces and generalized inertia forces equal to zero, which produces

$$\begin{aligned} M_{ur}\ddot{q}_r - [(2^{-1}M_{rs,u} - M_{ur,s})\dot{q}_s + \boldsymbol{\omega} \cdot (\mathbf{h}_{u,r} - \mathbf{h}_{r,u})]\dot{q}_r \\ + \dot{\boldsymbol{\omega}} \cdot \mathbf{h}_u - 2^{-1}\boldsymbol{\omega} \cdot \mathbf{I}_{,u} \cdot \boldsymbol{\omega} + V_{,u} - D_u = 0 \quad u = 1, \ldots, n \end{aligned} \tag{19}$$

These equations will be referred to as the *internal equations of motion.*

QUASI-RIGID BODY MOTION

A motion of S during which $q_1, \ldots, q_n$ remain equal to zero will be called a quasi-rigid body motion. During a quasi-rigid body motion, the external equations of motion, (16), reduce to

$$\mathbf{n}_i \cdot \tilde{\mathbf{I}} \cdot \boldsymbol{\omega} = H_i \quad i = 1, 2, 3 \tag{20}$$

while the internal equations of motion, (19), become

$$\dot{\boldsymbol{\omega}} \cdot \tilde{\mathbf{h}}_u - 2^{-1}\boldsymbol{\omega} \cdot \tilde{\mathbf{I}}_{,u} \cdot \boldsymbol{\omega} + \tilde{V}_{,u} = 0 \quad u = 1, \ldots, n \tag{21}$$

Here (3) has been used to eliminate $D_1, \ldots, D_n$ from the internal equations of motion and, as throughout the sequel, a tilde over a symbol indicates that the quantity represented by the symbol is to be evaluated at $q_1 = \ldots = q_n = 0$; for example, $\tilde{\mathbf{h}}_1 = \mathbf{h}_1(0)$.

Under certain circumstances, a quasi-rigid body motion is possible when $\boldsymbol{\omega}$ has a constant magnitude Ω and is at all times parallel to $\mathbf{b}_1$; that is, $\boldsymbol{\omega}$ remains fixed in B and hence also in N. Such a motion is called a *simple spin.* Conditions sufficient for the existence of a simple spin can be obtained from (20) and (21): replacing $\boldsymbol{\omega}$ with $\Omega\mathbf{b}_1$ in (20) and (21) yields

$$\Omega \tilde{I}_1 \mathbf{b}_1 \cdot \mathbf{n}_i = H_i \quad i = 1, 2, 3 \tag{22}$$

and

$$-2^{-1}\Omega^2 \mathbf{b}_1 \cdot \tilde{\mathbf{I}}_{,u} \cdot \mathbf{b}_1 + \tilde{V}_{,u} = 0 \quad u = 1, \ldots, n \tag{23}$$

respectively, where $\tilde{I}_1$ is the moment of inertia of S about B_1 when $q_1 = \cdots = q_n = 0$. (22) is identically satisfied during a simple spin and satisfaction of (23) guarantees the existence of a simple spin.

LINEARIZED EQUATIONS

Together, (16) and (19) form a set of $n+3$ differential equations in the unknowns ω_1, ω_2, ω_3, $q_1, \ldots, q_n$. The coefficients of these quantities and of their time derivatives are functions of $q_1, \ldots, q_n$ since $\mathbf{p}$ is such a function and the coefficients all depend on $\mathbf{p}$ [see (10) and (11)]. Consequently, the equations of motion are nonlinear. Moreover, they are strongly coupled. Hence, one is confronted with formidable obstacles when attempting to solve these equations. However, the following possibility presents itself: if the system under consideration possesses a quasi-rigid body motion, one may hope to extract information about nearly quasi-rigid body motions from equations generated by linearizing the present equations in $q_1, \ldots, q_n$; and the new equations may be more manageable than the earlier ones. This turns out to be the case, as will now be shown.

To begin the linearization, $\mathbf{I}$, $\mathbf{h}_u$, M_{ur}, $V_{,u}$, and D_u are expanded in Taylor series:

$$\mathbf{I} = \tilde{\mathbf{I}} + \tilde{\mathbf{I}}_{,r} q_r + 2^{-1} q_r \tilde{\mathbf{I}}_{,rs} q_s + O_3(q) \tag{24}$$

$$\mathbf{h}_u = \tilde{\mathbf{h}}_u + \tilde{\mathbf{h}}_{u,r} q_r + O_2(q) \quad u = 1, \ldots, n \tag{25}$$

$$M_{ur} = \tilde{M}_{ur} + O_1(q) \quad r, u = 1, \ldots, n \tag{26}$$

$$V_{,u} = \tilde{V}_{,u} + \tilde{V}_{,ur} q_r + O_2(q) \quad u = 1, \ldots, n \tag{27}$$

$$D_u = \tilde{D}_{u,r} q_r + \tilde{D}_{u,\dot{r}} \dot{q}_r + O_2(q, \dot{q}) \quad u = 1, \ldots, n \tag{28}$$

where, for example, $O_3(q)$ indicates terms of third or higher degree in $q_1, \ldots, q_n$. Next, these expressions are substituted into (19), and all terms of second or higher degree in $q_1, \ldots, q_n$ are dropped. Thus one arrivers at the *linearized internal equations of motion*

$$\begin{aligned} &\tilde{M}_{ur}\ddot{q}_r + [\boldsymbol{\omega} \cdot (\tilde{\mathbf{h}}_{u,r} - \tilde{\mathbf{h}}_{r,u}) - \tilde{D}_{u,\dot{r}}]\dot{q}_r \\ &\quad + (\dot{\boldsymbol{\omega}} \cdot \tilde{\mathbf{h}}_{u,r} - 2^{-1}\boldsymbol{\omega} \cdot \tilde{\mathbf{I}}_{,ur} \cdot \boldsymbol{\omega} + \tilde{V}_{,ur} - \tilde{D}_{u,r})q_r \\ &= -\dot{\boldsymbol{\omega}} \cdot \tilde{\mathbf{h}}_u + 2^{-1}\boldsymbol{\omega} \cdot \tilde{\mathbf{I}}_{,u} \cdot \boldsymbol{\omega} - \tilde{V}_{,u} \quad u = 1, \ldots, n \end{aligned} \tag{28}$$

Before proceeding to the linearization of the external equations of motion, it is convenient to define matrices ω, C, I, $I_{,r}$, h_r, and H as follows:

$$\omega \triangleq [\omega_1 \quad \omega_2 \quad \omega_3] \tag{30}$$

$$C \triangleq \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \tag{31}$$

where

$$C_{ij} \triangleq \mathbf{n}_i \cdot \mathbf{b}_j \quad i, j = 1, 2, 3 \tag{32}$$

$$I \triangleq \begin{bmatrix} \tilde{I}_1 & 0 & 0 \\ 0 & \tilde{I}_2 & 0 \\ 0 & 0 & \tilde{I}_3 \end{bmatrix} \tag{33}$$

where

$$\tilde{I}_i \triangleq \mathbf{b}_i \cdot \tilde{\mathbf{I}} \cdot \mathbf{b}_i \quad i = 1, 2, 3 \tag{34}$$

$$I_{,r} \triangleq \begin{bmatrix} \tilde{I}_{1,r} & \tilde{I}_{12,r} & \tilde{I}_{13,r} \\ \tilde{I}_{21,r} & \tilde{I}_{2,r} & \tilde{I}_{23,r} \\ \tilde{I}_{31,r} & \tilde{I}_{32,r} & \tilde{I}_{3,r} \end{bmatrix} \qquad r = 1, \ldots, n \tag{35}$$

where

$$\tilde{I}_{ij,r} \triangleq \mathbf{b}_i \cdot \tilde{\mathbf{I}}_{,r} \cdot \mathbf{b}_j \quad i, j = 1, 2, 3; \quad r = 1, \ldots, n \tag{36}$$

$$h_r \triangleq [\tilde{\mathbf{h}}_r \cdot \mathbf{b}_1 \;\; \tilde{\mathbf{h}}_r \cdot \mathbf{b}_2 \;\; \tilde{\mathbf{h}}_r \cdot \mathbf{b}_3] \quad r = 1, \ldots, n \tag{37}$$

and

$$H \triangleq [H_1 \quad H_2 \quad H_3] \tag{38}$$

These, together with (24) and (25) and the fact that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are parallel to principal axes of S for S^* when $q_1 = \cdots = q_n = 0$, permit one to rewrite (16) in the matrix form

$$[\omega\{I + I_{,r}q_r + O_2(q)\} + \{h_r\dot{q}_r + O_2(q, \dot{q})\}]C^T = H \tag{39}$$

where C^T denotes the transpose of C. Furthermore, since C^TC equals the identy matrix, one can solve (39) for ω, obtaining

$$\omega = [HC - h_r\dot{q}_r - O_2(q, \dot{q})]\,[I + I_{,r}q_r + O_2(q)]^{-1} \tag{40}$$

The matrix inverse appearing in (40) can be evaluated as follows:

$$[I + I_{,r}q_r + O_2(q)]^{-1} = \Delta^{-1}\begin{bmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} \end{bmatrix} \tag{41}$$

where Δ represents the determinant of $[I + I_{,r}q_r + O_2(Q)]$ and Δ_{ij} $(i, j = 1, 2, 3)$ is the cofactor of the element in the ith row and jth column of $[I + I_{,r}q_r + O_2(q)]$. After evaluating the cofactors, one has

$$[I + I_{,r}q_r + O_2(q)]^{-1} =$$

$$\Delta^{-1}\begin{bmatrix} \tilde{I}_2\tilde{I}_3 + (\tilde{I}_2\tilde{I}_{3,r} + \tilde{I}_{2,r}\tilde{I}_3)q_r & -\tilde{I}_3\tilde{I}_{12,r}q_r & -\tilde{I}_2\tilde{I}_{13,r}q_r \\ -\tilde{I}_3\tilde{I}_{21,r}q_r & \tilde{I}_3\tilde{I}_1 + (\tilde{I}_3\tilde{I}_{1,r} + \tilde{I}_{3,r}\tilde{I}_1)q_r & -\tilde{I}_1\tilde{I}_{23,r}q_r \\ -\tilde{I}_2\tilde{I}_{31,r}q_r & -\tilde{I}_1\tilde{I}_{32,r}q_r & \tilde{I}_1\tilde{I}_2 + (\tilde{I}_1\tilde{I}_{2,r} + \tilde{I}_{1,r}\tilde{I}_2)q_r \end{bmatrix} + O_2 \tag{42}$$

The determinant Δ can be written as

$$\Delta = \tilde{I}_1\tilde{I}_2\tilde{I}_3 + (\tilde{I}_1\tilde{I}_2\tilde{I}_{3,r} + \tilde{I}_1\tilde{I}_{2,r}\tilde{I}_3 + \tilde{I}_{1,r}\tilde{I}_2\tilde{I}_3)q_r + O_2(q) \tag{43}$$

and use of the binomial theorem permits one to write

$$\Delta^{-1} = (\tilde{I}_1\tilde{I}_2\tilde{I}_3)^{-1}[1 - \tilde{I}_{i,r}\tilde{I}_i^{-1}q_r + O_2(q)]. \tag{44}$$

One can now eliminate Δ from (42), with the result

$$[I + I_{,r}q_r + O_2(q)]^{-1} = \begin{bmatrix} \tilde{I}_1^{-1}(1 - \tilde{I}_1^{-1}\tilde{I}_{1,r}q_r) & -(\tilde{I}_1\tilde{I}_2)^{-1} \times \tilde{I}_{12,r}q_r & -(\tilde{I}_1\tilde{I}_3)^{-1} \times \tilde{I}_{13,r}q_r \\ -(\tilde{I}_2\tilde{I}_1)^{-1} \times \tilde{I}_{21,r}q_r & \tilde{I}_2^{-1}(1 - \tilde{I}_2^{-1}\tilde{I}_{2,r}q_r) & -(\tilde{I}_2\tilde{I}_3)^{-1} \times \tilde{I}_{23,r}q_r \\ -(\tilde{I}_3\tilde{I}_1)^{-1} \times \tilde{I}_{31,r}q_r & -(\tilde{I}_3\tilde{I}_2)^{-1} \times \tilde{I}_{32,r}q_r & \tilde{I}_3^{-1}(1 - \tilde{I}_3^{-1}\tilde{I}_{3,r}q_r) \end{bmatrix} + O_2 \tag{45}$$

Substitution from (45) into (40) and subsequent linearization in $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n$ leads to the *linearized external equation of motion*

$$\omega = HC \begin{bmatrix} \tilde{I}_1^{-1}(1-\tilde{I}_1^{-1}\tilde{I}_{1,r}q_r) & -(\tilde{I}_1\tilde{I}_2)^{-1}\times\tilde{I}_{12,r}q_r & -(\tilde{I}_1\tilde{I}_3)^{-1}\times\tilde{I}_{13,r}q_r \\ -(\tilde{I}_2\tilde{I}_1)^{-1}\times\tilde{I}_{21,r}q_r & \tilde{I}_2^{-1}(1-\tilde{I}_2^{-1}\tilde{I}_{2,r}q_r) & -(\tilde{I}_2\tilde{I}_3)^{-1}\times\tilde{I}_{23,r}q_r \\ -(\tilde{I}_3\tilde{I}_1)^{-1}\times\tilde{I}_{31,r}q_r & -(\tilde{I}_3\tilde{I}_2)^{-1}\times\tilde{I}_{32,r}q_r & \tilde{I}_3^{-1}(1-\tilde{I}_3^{-1}\tilde{I}_{3,r}q_r) \end{bmatrix} - h_r\dot{q}_r \begin{bmatrix} \tilde{I}_1^{-1} & 0 & 0 \\ 0 & \tilde{I}_2^{-1} & 0 \\ 0 & 0 & \tilde{I}_3^{-1} \end{bmatrix} \tag{46}$$

The matrix C appearing in this equation requires further consideration. Since the elements of C depend on the orientation of the unit vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ relative to the unit vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ [see (32)], that is, on the external variables θ_1, θ_2, θ_3, the matrix C is, in fact, an unknown matrix. However, it can be related to ω through a set of so-called kinematical equations, one form of which involves Euler parameters $\epsilon_1, \ldots, \epsilon_4$. In terms of these, C_{ij} $(i, j = 1, 2, 3)$ are given by (see [28], p. 8)

$$\left.\begin{aligned} C_{11} &= 1-2(\epsilon_2^2+\epsilon_3^2) \\ C_{12} &= 2(\epsilon_1\epsilon_2-\epsilon_3\epsilon_4) \\ C_{13} &= 2(\epsilon_3\epsilon_1+\epsilon_2\epsilon_4) \\ C_{21} &= 2(\epsilon_1\epsilon_2+\epsilon_3\epsilon_4) \\ C_{22} &= 1-2(\epsilon_3^2+\epsilon_1^2) \\ C_{23} &= 2(\epsilon_2\epsilon_3-\epsilon_4\epsilon_1) \\ C_{31} &= 2(\epsilon_3\epsilon_1-\epsilon_2\epsilon_4) \\ C_{32} &= 2(\epsilon_2\epsilon_3+\epsilon_4\epsilon_1) \\ C_{33} &= 1-2(\epsilon_1^2+\epsilon_2^2) \end{aligned}\right\} \tag{47}$$

and $\epsilon_1, \ldots, \epsilon_4$ are related to ω through the differential equation

$$2[\dot{\epsilon}_1 \quad \dot{\epsilon}_2 \quad \dot{\epsilon}_3 \quad \dot{\epsilon}_4] = \omega \begin{bmatrix} \epsilon_4 & \epsilon_3 & -\epsilon_2 & -\epsilon_1 \\ -\epsilon_3 & \epsilon_4 & \epsilon_1 & -\epsilon_2 \\ \epsilon_2 & -\epsilon_1 & \epsilon_4 & -\epsilon_3 \end{bmatrix} \tag{48}$$

The right-hand member of this equation can be expressed as a function of $q_1, \ldots, q_n$, $\dot{q}_1, \ldots, \dot{q}_n$, and $\epsilon_1, \ldots, \epsilon_4$ by using (46) to eliminate ω. The resulting matrix equation furnishes four differential equations which, together with n equations derived from (29), ultimately permit one of determine the configuration of S in N by solving for the $n+4$ quantities $q_1, \ldots, q_n$, and $\epsilon_1, \ldots, \epsilon_4$. To obtain the latter set of n equations, one first forms an expression for $\dot{\omega}$ by differentiating (46), then eliminates $\dot{\omega}$ and ω from (29) by using this expression and (46), and finally linearizes in $q_1, \ldots, q_n$, $\dot{q}_1, \ldots, \dot{q}_n$. The result can be expressed in a convenient form in terms of three n-dimensional square matrices U, X, Y and two n-dimensional column matrices Z and Q with elements U_{ur}, X_{ur}, Y_{ur}, Z_u, Q_u, respectively, defined as

$$U_{ur} \triangleq \tilde{M}_{ur} - \tilde{I}_i^{-1}\tilde{\mathbf{h}}_r \cdot \mathbf{b}_i \tilde{\mathbf{h}}_u \cdot \mathbf{b}_i \tag{49}$$

$$X_{ur} \triangleq H_i C_{ij} \tilde{I}_j^{-1}[\tilde{I}_k^{-1}(\tilde{I}_{jk,r}\tilde{\mathbf{h}}_u - \tilde{I}_{jk,u}\tilde{\mathbf{h}}_r) \cdot \mathbf{b}_k + (\tilde{\mathbf{h}}_{r,u} - \tilde{\mathbf{h}}_{u,r}) \cdot \mathbf{b}_j] + \tilde{D}_{u,\dot{r}} \tag{50}$$

$$\begin{aligned} Y_{ur} \triangleq {} & H_i \dot{C}_{ij} \tilde{I}_j^{-1}(\tilde{I}_{jk,r}\tilde{I}_k^{-1}\tilde{\mathbf{h}}_u \cdot \mathbf{b}_k - \tilde{\mathbf{h}}_{u,r} \cdot \mathbf{b}_j) + H_i C_{ij} \tilde{I}_j^{-1}\tilde{I}_l^{-1} H_m C_{ml} \\ & (2^{-1}\tilde{I}_{jl,ur} - \tilde{I}_{jk,r}\tilde{I}_{kl,u}\tilde{I}_k^{-1}) - \tilde{V}_{,ur} + \tilde{D}_{u,r} \end{aligned} \tag{51}$$

$$Z_u \triangleq -H_i \dot{C}_{ij} \tilde{I}_j^{-1}\tilde{\mathbf{h}}_u \cdot \mathbf{b}_j + 2^{-1} H_i C_{ij} \tilde{I}_{jl,u} \tilde{I}_j^{-1}\tilde{I}_l^{-1} H_m C_{ml} - \tilde{V}_{,u} \tag{52}$$

$$Q_u \triangleq q_u \tag{53}$$

Specifically, one finds that

$$\ddot{Q} = U^{-1}(X\dot{Q} + YQ + Z) \tag{54}$$

where U^{-1} is the inverse of U.

SIMPLIFICATIONS

Implicitly, (48) and (54) depend upon the quantitities $\tilde{M}_{ur}$, $\tilde{\mathbf{h}}_u$, $\tilde{\mathbf{h}}_{u,r}$, $\tilde{\mathbf{I}}$, $\tilde{\mathbf{I}}_{,u}$, $\tilde{\mathbf{I}}_{,ur}$ $(r, u = 1, \ldots, n)$ each of which involves a summation over all particles of S [see (10) and (11)]. If a subset of S forms a rigid body R, then the contribution of R to these quantities can be written in forms not explicitly involving sums over the particles of R, but involving, instead, inertia properties of R. For example, $(\tilde{M}_{ur})_R$, the contribution to $\tilde{M}_{ur}$ of R, is given by

$$(\tilde{M}_{ur})_R = m_R \tilde{\mathbf{p}}^{R^*}_{,u} \cdot \tilde{\mathbf{p}}^{R^*}_{,r} + {}^B\tilde{\boldsymbol{\omega}}^R_{,\dot{u}} \cdot \tilde{\mathbf{I}}^{R/R^*} \cdot {}^B\tilde{\boldsymbol{\omega}}^R_{,\dot{r}} \quad r, u = 1, \ldots, n \tag{55}$$

where m_R is the mass of R; $\mathbf{p}^{R^*}$ is the position vector of R^*, the mass center of R, relative to S^*; ${}^B\boldsymbol{\omega}^R_{,\dot{u}}$ $(u = 1, \ldots, n)$ is the partial rate of change of orientation of R with respect to q_u in B; and $\mathbf{I}^{R/R^*}$ is the inertia dyadic of R for R^* [see [29], p. 140, eqn. (4.26)]. To generate analogous expressions for $(\tilde{\mathbf{h}}_u)_R$ and $(\tilde{\mathbf{h}}_{u,r})_R$ $(r, u = 1, \ldots, n)$, the contributions to $\tilde{\mathbf{h}}_u$ and $\tilde{\mathbf{h}}_{u,r}$ of R, respectively, let $\mathbf{H}^{R/S^*}$ denote the angular momentum of R in B relative to S^*. Then

$$\mathbf{H}^{R/S^*} \stackrel{\Delta}{=} \langle m\,\mathbf{p} \times \mathbf{v} \rangle \tag{56}$$

where now the symbols $\langle\,\rangle$ imply summation over all particles of R, and m, $\mathbf{p}$, and $\mathbf{v}$ are respectively the mass, the position vector relative to S^*, and the velocity in B of a generic particle of R. Expressing $\mathbf{v}$ in terms of derivatives of $\mathbf{p}$ gives

$$\mathbf{H}^{R/S^*} = \langle m\,\mathbf{p} \times \mathbf{p}_{,u} \rangle \dot{q}_u \underset{(11)}{=} (\mathbf{h}_u)_R \dot{q}_u \tag{57}$$

where $(\mathbf{h}_u)_R$ $(u = 1, \ldots, n)$ represents the contribution of R to $\mathbf{h}_u$.

If $\mathbf{H}^{R/R^*}$ denotes the angular momentum of R in B relative to R^*, and $\mathbf{H}^{R^*/S^*}$ denotes the angular momentum in B relative to S^* of a fictitious particle whose motion is identical to that of R^* and whose mass is equal to m_R, then

$$\mathbf{H}^{R/S^*} = \mathbf{H}^{R/R^*} + \mathbf{H}^{R^*/S^*} \tag{58}$$

where

$$\mathbf{H}^{R/R^*} = \mathbf{I}^{R/R^*} \cdot ({}^B\boldsymbol{\omega}^R_{,\dot{u}} \dot{q}_u) \tag{59}$$

and

$$\mathbf{H}^{R^*/S^*} = m_R \mathbf{p}^{R^*} \times (\mathbf{p}^{R^*}_{,u} \dot{q}_u) \tag{60}$$

(see [29], p. 163, problem 7l). Using (57), (59), (60) to eliminate $\mathbf{H}^{R/S^*}$, $\mathbf{H}^{R/R^*}$, and H^{R^*/S^*}, from (58), subsequently equating the coefficients of $\dot{q}_u$ $(u = 1, \ldots, n)$ to zero, and finally solving for $(\mathbf{h}_u)_R$ yields

$$(\mathbf{h}_u)_R = \mathbf{I}^{R/R^*} \cdot {}^B\boldsymbol{\omega}^R_{,\dot{u}} + m_R \mathbf{p}^{R^*} \times \mathbf{p}^{R^*}_{,u} \quad u = 1, \ldots, n. \tag{61}$$

Differentiating with respect to q_r $(r = 1, \ldots, n)$ and letting $q_1 = \ldots, q_n = 0$ produces the desired result, namely

$$(\tilde{\mathbf{h}}_{u,r})_R = \tilde{\mathbf{I}}^{R/R^*}_{,r} \cdot {}^B\tilde{\boldsymbol{\omega}}^R_{,\dot{u}} + \tilde{\mathbf{I}}^{R/R^*} \cdot {}^B\tilde{\boldsymbol{\omega}}^R_{,\dot{u}r} + m_R(\tilde{\mathbf{p}}^{R^*}_{,r} \times \tilde{\mathbf{p}}^{R^*}_{,u} + \tilde{\mathbf{p}}^{R^*} \times \tilde{\mathbf{p}}^{R^*}_{,ur}) \quad r, u = 1, \ldots, n \tag{62}$$

while letting $q_1 = \cdots = q_n = 0$ in (61) yields

$$(\tilde{\mathbf{h}}_u)_R = \tilde{\mathbf{I}}^{R/R^*} \cdot {}^B\tilde{\boldsymbol{\omega}}^R_{,u} + m_R \tilde{\mathbf{p}}^{R^*} \times \tilde{\mathbf{p}}^{R^*}_{,u} \quad u = 1, \ldots, n \tag{63}$$

The contributions of R to $\tilde{\mathbf{I}}$, $\tilde{\mathbf{I}}_{,u}$, and $\tilde{\mathbf{I}}_{,ur}$ $(r, u = 1, \ldots, n)$ are found most conveniently by expressing $(\tilde{I}_i)_R$, $(\tilde{I}_{ij,u})_R$, and $(\tilde{I}_{ij,ur})_R$, defined respectively as the contributions of R to the elements of $\tilde{\mathbf{I}}$, $\tilde{\mathbf{I}}_{,u}$ and $\tilde{\mathbf{I}}_{,ur}$, in terms of I^{R/R^*}_{ij} $(i, j = 1, 2, 3)$, the product of inertia of R relative to R^* for $\mathbf{r}_i$

and $\mathbf{r}_j$, where $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ form a dextral set of orthogonal unit vectors fixed in R, and β_{ij}^R $(i, j = 1, 2, 3)$, defined as

$$\beta_{ij}^R \triangleq \mathbf{b}_i \cdot \mathbf{r}_j \quad i, j = 1, 2, 3 \tag{64}$$

The desired relations are

$$(\tilde{I}_i)_R = \tilde{\beta}_{ik}^R I_{kl}^{R/R^*} \tilde{\beta}_{il}^R + m_R[(\tilde{\mathbf{p}}^{R^*})^2 - (\tilde{\mathbf{p}}^{R^*} \cdot \mathbf{b}_i)^2] \quad i = 1, 2, 3 \tag{65}$$

$$\begin{aligned}(\tilde{I}_{ij,u})_R = {} & I_{kl}^{R/R^*}(\tilde{\beta}_{ik,u}^R \tilde{\beta}_{jl}^R + \tilde{\beta}_{ik}^R \tilde{\beta}_{jl,u}^R) + m_R[2(\tilde{\mathbf{p}}^{R^*} \cdot \tilde{\mathbf{p}}_{,u}^{R^*})(\mathbf{b}_i \cdot \mathbf{b}_j) - (\tilde{\mathbf{p}}_{,u}^{R^*} \cdot \mathbf{b}_j)(\tilde{\mathbf{p}}^{R^*} \cdot \mathbf{b}_i) \\ & - (\tilde{\mathbf{p}}_{,u}^{R^*} \cdot \mathbf{b}_i)(\tilde{\mathbf{p}}^{R^*} \cdot \mathbf{b}_j)] \quad i, j = 1, 2, 3; \quad u = 1, \ldots, n\end{aligned} \tag{66}$$

$$\begin{aligned}(\tilde{I}_{ij,ur})_R = {} & I_{kl}^{R/R^*}(\tilde{\beta}_{ik,ur}^R \tilde{\beta}_{jl}^R + \tilde{\beta}_{ik,u}^R \tilde{\beta}_{jl,r}^R + \tilde{\beta}_{ik,r}^R \tilde{\beta}_{jl,u}^R + \tilde{\beta}_{ik}^R \tilde{\beta}_{jl,ur}^R) \\ & + m_R[2(\tilde{\mathbf{p}}_{,ur}^{R^*} \cdot \tilde{\mathbf{p}}^{R^*} + \tilde{\mathbf{p}}_{,u}^{R^*} \cdot \tilde{\mathbf{p}}_{,r}^{R^*}) \times (\mathbf{b}_i \cdot \mathbf{b}_j) - (\tilde{\mathbf{p}}_{,ur}^{R^*} \cdot \mathbf{b}_i)(\tilde{\mathbf{p}}^{R^*} \cdot \mathbf{b}_j) - (\tilde{\mathbf{p}}_{,ur}^{R^*} \cdot \mathbf{b}_j)(\tilde{\mathbf{p}}^{R^*} \cdot \mathbf{b}_i) \\ & - (\tilde{\mathbf{p}}_{,r}^{R^*} \cdot \mathbf{b}_i) \times (\tilde{\mathbf{p}}_{,u}^{R^*} \cdot \mathbf{b}_j) - (\tilde{\mathbf{p}}_{,r}^{R^*} \cdot \mathbf{b}_j)(\tilde{\mathbf{p}}_{,u}^{R^*} \cdot \mathbf{b}_i) \quad i, j = 1, 2, 3; \quad r, u = 1, \ldots, n\end{aligned} \tag{67}$$

Any particle of S that does not belong to a rigid subset of S can be considered as a rigid body with zero inertia dyadic $\mathbf{I}^{R/R^*}$. Thus, one can always regard S as a collection of γ rigid bodies and use each of (55), (62), (63), and (65)–(67) γ times to evaluate the contributions of the γ bodies to $\tilde{M}_{ur}$, $\tilde{\mathbf{h}}_u$, $\tilde{\mathbf{h}}_{u,r}$, $\tilde{I}_i$, $\tilde{I}_{ij,u}$, $\tilde{I}_{ij,ur}$ $(i, j = 1, 2, 3; r, u = 1, \ldots, n)$, and this makes it unnecessary ever to carry out a summation over the particles of S.

COMPUTER PROGRAM SAM

(48) and (54) form the basis for the FORTRAN computer program SAM (Satellite Attitude Motions). This program is intended to be a tool an analyst can use to obtain information about the motion of a dynamical system without deriving equations of motion and without writing any computer code. Rather, the analyst furnishes a set of constants characterizing certain properties of the system in its undeformed state, a set of initial values for the internal coordinates and their time derivatives, the initial values of the $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ measure numbers of the angular velocity of B in N, four integration parameters, a title, and a date; the program then produces a detailed description of the motion that occurs subsequent to the initial instant.

The analytical tasks that must be performed by the user of the program begin with the numbering of the bodies forming the system S and the internal coordinates of S as, respectively, $1, \ldots, \gamma$ and $1, \ldots, n$, the internal coordinates having been selected in such a way that they may be expected to remain small throughout some time interval beginning with the initial instant. Next, to account for the distribution of mass throughout S, a mass m_R is assigned to body R $(R = 1, \ldots, \gamma)$, an arbitrary reference point O of S is selected, and the position vector $\mathbf{g}^R$ of the mass center of body R relative to O is used to express g_i^R, defined as $g_i^R \triangleq \mathbf{g}^R \cdot \mathbf{b}_i$, as a function of $q_1, \ldots, q_n$, where $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are unit vectors forming a dextral, orthogonal set such that each unit vector is parallel to a principal axis of inertia of S for the mass center S^* of S when $q_1, \ldots, q_n$ all vanish. These functions are then evaluated at $q_1 = \cdots = q_n = 0$ to obtain quantities designated $\tilde{g}_i^R$ $(i = 1, 2, 3; R = 1, \ldots, \gamma)$, and they are differentiated with respect to the internal coordinates to generate first derivatives $\tilde{g}_{i,r}^R$ and second derivatives $\tilde{g}_{i,rs}^R$ $(i = 1, 2, 3;\ r, s = 1, \ldots, n;$ $R = 1, \ldots, \gamma)$, where tildes again denote evaluation at $q_1 = \cdots = q_n = 0$. Similarly, to deal with moments and products of inertia of S, $\mathbf{I}^{R/R^*}$, the central inertia dyadic of body R, and $\mathbf{r}_1^R$, $\mathbf{r}_2^R$, $\mathbf{r}_3^R$, a dextral set of orthogonal unit vectors fixed in body R, are employed to form I_{jk}^{R/R^*}, defined as $I_{jk}^{R/R^*} \triangleq \mathbf{r}_j^R \cdot \mathbf{I}^{R/R^*} \cdot \mathbf{r}_k^R$; and required direction cosine information related to body R is generated by defining β_{ij}^R as $\beta_{ij}^R \triangleq \mathbf{b}_i \cdot \mathbf{r}_j^R$, expressing β_{ij}^R as a function of $q_1, \ldots, q_n$, and then evaluating $\tilde{\beta}_{ij}^R$, $\tilde{\beta}_{ij,r}^R$ and $\tilde{\beta}_{ij,rs}^R$ $(i, j = 1, 2, 3;\ r, s = 1, \ldots, n;\ R = 1, \ldots, \gamma)$. Here, if body R is a particle, all quantities involving R as a superscript are set equal to zero. The final task related to the individual bodies forming S is to let ${}^B\boldsymbol{\omega}^R$ be the angular velocity of body R in a reference frame B in which $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are fixed, to define ω_i^R as $\omega_i^R \triangleq {}^B\boldsymbol{\omega}^R \cdot \mathbf{b}_i$, and to evaluate the derivatives $\tilde{\omega}_{i,r}^R$ and $\tilde{\omega}_{i,rs}^R$ ($i = 1, 2, 3$; $r, s = 1, \ldots, n$; $R = 1, \ldots \gamma$), again setting equal to zero all quantities involving R as a superscript if body R is a particle. It then only remains to formulate $V(q_1, \ldots, q_n)$ and $D_r(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$, which must be such that the generalized active force F_r corresponding to q_r can be expressed as $F_r = -V_{,r} + D_r$, and to evaluate the derivatives $\tilde{V}_{,r}$, $\tilde{V}_{,rs}$, $\tilde{D}_{r,s}$, and $\tilde{D}_{r,\dot{s}}$ ($r, s = 1, \ldots, n$). Once all of this has been done, the analyst can assign initial values to q_r, $\dot{q}_r$ ($r = 1, \ldots, n$), and ω_i, defined as $\omega_i \triangleq \boldsymbol{\omega} \cdot \mathbf{b}_i$ ($i = 1, 2, 3$, where $\boldsymbol{\omega}$ is the angular velocity of B in N, select the interval of time for which a simulation of motion is desired, prepare data records, and make a computer run. (More detailed instructions for using SAM, as well as a listing of the FORTRAN code for SAM, are contained in [30].)

The integration algorithm used by SAM requires algebraic expressions for the highest derivatives of the dependent variables in terms of the lower derivatives of the dependent variables, the dependent variables themselves, and the independent variable. Thus expressions for $\dot{\epsilon}_i$ and $\ddot{q}_r$ in terms of ϵ_i, q_r, $\dot{q}_r$ ($i = 1, \ldots, 4$; $r = 1, \ldots, n$) are required in the present case. For $\dot{\epsilon}_1, \ldots, \dot{\epsilon}_4$, these are obtained from (48) after using (46) and (47) to eliminate ω and C; and (54), together with (49)–(53), furnish the necessary expressions for $\ddot{q}_1, \ldots, \ddot{q}_n$.

The output from SAM includes a list of all the input records, and, if errors are detected by SAM in the input data, error messages. In the event errors are detected in the form of the input data, the program terminates before beginning to integrate any equations. If no errors are detected in the input data, the output continues with a table containing values of time, the angle between $\mathbf{b}_1$ and the linearized angular momentum $\mathbf{H}$ (a vector fixed in N obtained by using (39) and dropping terms of second and higher degree in $q_1, \ldots, q_n$), the magnitude of the linearized angular momentum (which should remain constant), the $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ measure numbers of the angular velocity of B in N [see (46)], the Euler parameters relating $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ to unit vectors fixed in N and initially coincident with $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, the sum of the squares of the Euler parameters (which should remain equal to unity), the internal coordinates, and the first time derivatives of the internal coordinates.

EXAMPLE

To illustrate the use of the program SAM, a particular motion of a spacecraft carrying a passive nutation damper is simulated. The same system has been studied in detail by Kane and Levinson [31]. It is composed of a rigid body W and a particle P, as indicated in Fig. 1. W_1, W_2, W_3 are principal axes of inertia of W for W^*, the mass center of W, and $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ are unit vectors respectively parallel to W_1, W_2, W_3. W is assumed to be a homogeneous right parallelepiped with sides L_1, L_2, L_3 and mass m_W. The particle P has a mass m_p and is constrained to move on a line parallel to W_1 and passing through W_2 at a distance "a" from W^*. The particle P is attached to W by a massless linear spring and dashpot such that the force transmitted to P by W is given by $-(\sigma x + \mu \dot{x})\mathbf{w}_1$, where σ and μ are constants and x is the distance from W_2 to P. As before, N designates a Newtonian reference frame.

W may be numbered body 1, and P body 2. The system S formed by these bodies possesses but one internal degree of freedom, and the associated internal coordinate q_1 may be taken to be the distance x. S can perform a quasi-rigid body motion during which x remains equal to zero, namely a simple spin of W during which the angular velocity of W in N is equal to $\Omega \mathbf{w}_1$, where Ω is a constant. Hence, x may be expected to remain small, if not permanently, then at least throughout some time interval beginning at an instant at which, for example, $x = \dot{x} = 0$ and the angular velocity of W in N is given by $\mathbf{w}_1 + 0.1\mathbf{w}_2$. Next, values are assigned to m_1 and m_2 in accordance with the first line of Table 1; point W^* is selected as reference point 0; $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are taken to be $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, respectively; and g_i is expressed as shown in lines 2–4 of Table 1, after which $\tilde{g}_i^R$, $\tilde{g}_{i,r}^R$ and $\tilde{g}_{i,rs}^R$ are formed as shown in lines 5–13. Similarly, to deal with the inertia properties of S, $\mathbf{r}_1^R$, $\mathbf{r}_2^R$, $\mathbf{r}_3^R$ are taken to be $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, respectively, for $R = 1$ (for $R = 2$, inertia properties are not needed); I_{jk}^R and β_{ij}^R are expressed as shown in lines 14–19, and it is noted that $\tilde{\beta}_{ij,1}^R$ and $\tilde{\beta}_{ij,11}^R$ ($i, j = 1, 2, 3$; $R = 1, 2$) all vanish as do ω_i^R, and hence $\tilde{\omega}_{i,1}^R$ and $\tilde{\omega}_{i,11}^R$ ($i = 1, 2, 3$; $R = 1, 2$), for body 1 is fixed in B and body 2 is a particle. Thus, what remains to be done is to formulate $V(q_1)$ and $D_1(q_1, \dot{q}_1)$. Now, the generalized active force F_1 corresponding to q_1 is given by $F_1 = -(\sigma q_1 + \mu \dot{q}_1)$. Hence, if V and D_1 are taken to be $V = \sigma q_1^2/2$ and $D_1 = -\mu \dot{q}_1$, then $F_1 = -V_{,1} + D_1$, as required. Consequently, $\tilde{V}_{,1} = 0$, $\tilde{V}_{,11} = \sigma$, $\tilde{D}_{1,1} = 0$, and $\tilde{D}_{1,\dot{1}} = -\mu$.

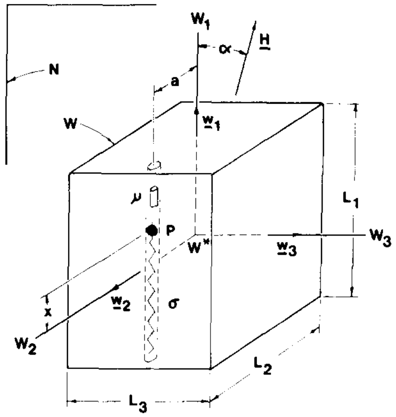

Fig. 1. Schematic representation of spacecraft with passive nutation damper.

Table 1. System constants for spacecraft with passive nutation damper

Line	Quantity	$R = 1$	$R = 2$
1	m_R	m_W	m_P
2	$g_1{}^R$	0	q_1
3	$g_2{}^R$	0	a
4	$g_3{}^R$	0	0
5	$\tilde{g}_1{}^R$	0	0
6	$\tilde{g}_2{}^R$	0	a
7	$\tilde{g}_3{}^R$	0	0
8	$\tilde{g}^R_{1,1}$	0	1
9	$\tilde{g}^R_{2,1}$	0	0
10	$\tilde{g}^R_{3,1}$	0	0
11	$\tilde{g}^R_{1,11}$	0	0
12	$\tilde{g}^R_{2,11}$	0	0
13	$\tilde{g}^R_{3,11}$	0	0
14	I^R_{11}	$m_W(L_2{}^2 + L_3{}^2)/12$	0
15	I^R_{22}	$m_W(L_3{}^2 + L_1{}^2)/12$	0
16	I^R_{33}	$m_W(L_1{}^2 + L_2{}^2)/12$	0
17	$I^R_{jk}(j \neq k)$	0	0
18	$\tilde{\beta}^R_{ij}\,(i = j)$	1	0
19	$\tilde{\beta}^R_{ij}\,(i \neq j)$	0	0

SAM was run with initial values and numerical values for the system parameters as follows: $q_1(0) = \dot{q}_1(0) = 0$, $\omega_1(0) = 1\ \text{rad sec}^{-1}$, $\omega_2(0) = 0.1\ \text{rad sec}^{-1}$, $\omega_3(0) = 0$, $a = 1$ m, $m_W = 5274.824$ kg, $m_P = 52.748$ kg, $L_1 = 2$ m, $L_2 = 0.948$ m, $L_3 = 1.008$ m, $\sigma = 52.748\ \text{Nm}^{-1}$, and $\mu = 105.496\ \text{N sec m}^{-1}$. The results permitted the construction of Fig. 2 which shows as the curve labeled "motion with nutation damper", a plot of α vs. t, where α is the angle between the angular momentum vector **H** and the unit vector $\mathbf{b}_1$ or, equivalently, the angle between **H** and line W_1 (see Fig. 1). In addition, Fig. 2 contains a line labeled "rigid body motion", which reveals how the angle between **H** and W_1 would vary with time if P were rigidly attached to W with $x = 0$. This line was constructed on the basis of the following considerations: When P is attached to W as stated, S is an axi-symmetric body; that is, S has the same moment of inertia about every line passing through the mass center of S and perpendicular to W_1 (the values for a, m_W, m_P, L_1, L_2, L_3 were chosen so as to produce this result). Consequently, the angle between W_1 and **H** must remain equal to its initial value throughout every motion of S.

The fact that the two curves in Fig. 2 become widely separated with increasing t justifies the conclusion that the particle P, despite its relatively small mass ($m_P/m_W = 0.01$), influences the motion of W profoundly. This is all the more noteworthy when one observes from the computer output of SAM that x varies in an oscillatory manner, never acquiring an absolute value in excess of 0.31 m during the time interval under consideration. But are these conclusions, in fact, valid? To answer this question, the exact differential equations of motion were integrated numerically. Gratifyingly, the agreement between the results of the numerical integration of exact equations and those obtained from SAM is very good. Indeed, the values of α obtained

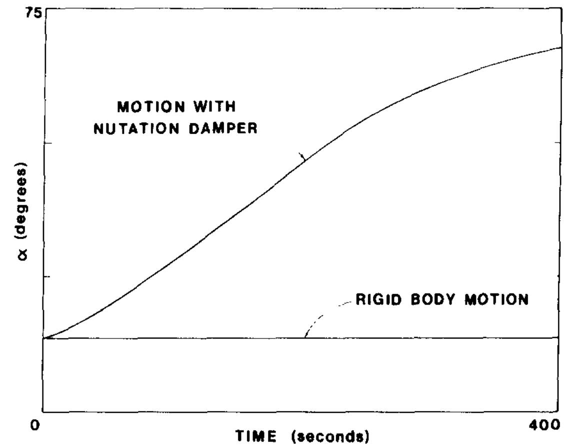

Fig. 2. Plot of α vs time for spacecraft with passive nutation damper.

from the two programs never differ from each other by more than 0.2 degrees. Fig. 2 thus represents the exact solution as well as the one produced by SAM, and one may conclude that SAM is capable of simulating motions involving large attitude changes, the linearizations underlying SAM notwithstanding.

The present example illustrates the practical utility of SAM in a number of ways. For example, it shows that SAM furnishes convenient means for testing the stability of a motion, for it will be recalled that W would perform a simple spinning motion if the initial conditions were modified in only one respect, namely by changing $\omega_2(0)$ from 0.1 rad/sec to zero. Thus the initial conditions used in the computer run represent a disturbance of the simple spin, and the results obtained indicate that the latter motion is unstable. The same conclusion was reached in [31] via a comparatively laborious stability analysis. However, the physical significance of an instability uncovered without simulation is always in doubt. Hence, even when a stability analysis has been performed, SAM can furnish useful additional information. Of course, this information could be gained also without using SAM, that is, by simulating the disturbed motion with a program based on exact equations of motion, as was done for the purpose of testing the validity of SAM in the present case. However, deriving exact equations of motion and programming their solution consumes considerably more time than does preparing the data records for SAM.

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